Correlations in an expanding gas of hard-core bosons

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We consider a longitudinal expansion of a one-dimensional gas of hard-core bosons suddenly released from a trap. We show that the broken translational invariance in the initial state of the system is encoded in correlations between the bosonic occupation numbers in the momentum space. The correlations are protected by the integrability and exhibit no relaxation during the expansion.

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Rapid progress in the ability to manipulate ultracold atomic gases stimulated a revival of interest in fundamental properties of interacting Bose systems. Unlike conventional condensed matter systems, cold gases offer a unique possibility to monitor the out-of-equilibrium dynamics of interacting systems unhindered by coupling to the environment and associated with it decoherence, see [1] for a recent review. Among various realizations of interacting Bose systems, the one-dimensional (1D) ones [2, 3, 4] occupy a special place: interactions in 1D have a much stronger effect than in higher dimensions while often allowing for a complete theoretical treatment.

This paper is partially motivated by the recent experimental study [3] of the evolution of 1D strongly interacting Bose liquid from a carefully prepared nonequilibrium initial state. The experiment [3] showed that the momentum distribution function does not exhibit a noticeable relaxation towards equilibrium. Such extremely slow relaxation is consistent with the behavior expected for an almost integrable system. Indeed, in a 1D system only three-particle collisions may lead to a momentum relaxation; such processes are absent in integrable models [5].

On the theoretical side, experiments such as [2, 3] highlight the relevance of the well-known in statistical mechanics *quantum quench* problem: how to describe the evolution of a system from an arbitrary initial state (see, e.g., [6] and references therein). At present, there are very few exact results on such strongly nonequilibrium dynamics of interacting quantum systems. Various problems of this type arise naturally in the description of the experiments on trapped cold atomic gases. Indeed, by far the most popular technique today is to observe the expansion of a gas after a sudden release of the trap [1]. Although such experiments are obviously destructive, the time-of-flight imaging [1] allows one to study the real-time evolution of the bosonic occupation numbers in the momentum space. Importantly, not only the average occupation numbers (momentum distribution) but also the corresponding higher-order statistical moments (fluctuations) are accessible experimentally [7, 8]). Unlike the momentum distribution, the fluctuations are sensitive to the relaxation in the system.

We consider a simple yet realistic [2] example: expansion of a 1D gas of bosons with infinitely strong contact repulsion (hard-core bosons) suddenly released from a trap. We show that shortly after the trap release, bosonic occupation numbers reach their steady-state values. We derive an operator identity, see Eq. (20) below, that relates the bosonic occupation numbers in the steady state to the integrals of motion. The identity allows one to study all statistical moments of the bosonic occupation numbers, and we evaluate the second moment in a closed form. Correlations between the occupation numbers at different momenta reflect directly the broken translational invariance in the initial (trapped) state of the system.

To be specific, we assume that initially (at $t < 0$) the system is in a thermal equilibrium state of the Hamiltonian

$$H = H_0 + V_{\text{trap}}. \tag{1}$$

Here H_0 describes 1D hard-core Bose gas without confinement (see Eq. (4) below) and

$$V_{\text{trap}} = \int dx\, V(x)\rho(x) \tag{2}$$

describes the effect of the trap, with $\rho(x)$ being the local density operator. The trap potential (2) breaks translational invariance, hence it does not commute with the Hamiltonian H_0 that governs the dynamics after the trap release at $t = 0$.

The simplest description of the hard-core bosons is based on the Jordan-Wigner transformation

$$\psi(x) = \exp\left[i\pi\int_{-\infty}^{x} dy\,\rho(y)\right]\varphi(x), \tag{3}$$

where the operators ψ and φ correspond to fermions and bosons, respectively:

$$\left\{\psi(x),\psi^\dagger(y)\right\} = \left[\varphi(x),\varphi^\dagger(y)\right] = \delta(x-y).$$

The transformation (3) maps the hard-core bosons onto free spinless fermions [9],

$$H_0 = \int dx\,\psi^\dagger(x)\left[-\frac{1}{2m}\frac{\partial^2}{\partial x^2}\right]\psi(x). \tag{4}$$

At the same time, the local density operator [and, therefore, Eq. (2)], retains its form,

$$\rho(x) = \psi^\dagger(x)\psi(x) = \varphi^\dagger(x)\varphi(x).$$

Since the fermionic occupation numbers in the momentum space

$$n_k = \psi_k^\dagger \psi_k, \quad \psi_k = (2\pi)^{-1/2}\int dx\, e^{-ikx}\psi(x), \tag{5}$$

commute with H_0, the expectation values of n_k are independent of time,

$$\langle n_k\rangle_t = \langle n_k\rangle_0 = \text{const.} \tag{6}$$

Hereinafter

$$\langle \widehat{\mathcal{O}}\,\rangle_t = \langle e^{iH_0 t}\widehat{\mathcal{O}}\, e^{-iH_0 t}\rangle_0,$$

where $\langle \ldots\rangle_0 \equiv \langle \ldots\rangle_{t\to 0}$ denotes the thermal averaging with the initial Hamiltonian H, see Eq. (1).

The crucial for the following observation is that even though the evolution of the system at $t > 0$ is governed by the *translationally invariant* Hamiltonian H_0, the initial Hamiltonian H does not have this symmetry. Therefore, not only the diagonal in k fermionic bilinears (such as n_k) have finite expectation values, but also the off-diagonal ones, e.g., $\langle \psi_k^\dagger \psi_{k'}\rangle_0 \neq 0$. Since the expectation value

$$\langle \psi_k^\dagger \psi_{k'}\rangle_t = \langle \psi_k^\dagger \psi_{k'}\rangle_0\, e^{i(\epsilon_k - \epsilon_{k'})\,t}, \quad \epsilon_p = p^2/2m, \tag{7}$$

oscillates with t, quantities such as $\left|\langle \psi_k^\dagger \psi_{k'}\rangle_t\right|$ remain constant and carry with them the memory of the broken-symmetry initial state of the system. Because the off-diagonal correlation functions (7) are finite, the fermionic occupation numbers fluctuate. Indeed, with the help of the Wick theorem one finds for $\delta n_k = n_k - \langle n_k\rangle_0$

$$\langle \delta n_k \delta n_{k'}\rangle_t = -\left|\langle \psi_k^\dagger \psi_{k'}\rangle_0\right|^2 = \text{const.} \tag{8}$$

It should be emphasized that the very survival of quantities such as Eqs. (7) and (8) that preserve the information about the initial conditions does not rely on the particularly simple form that the Hamiltonian H_0 has in our case. Rather, it is a direct consequence of the integrability. Indeed, in a generic (nonintegrable) system correlation function (7) would decay with t. This decay (relaxation) "washes out" the memory about the symmetry of the initial state, thereby restoring the translational invariance. After the relaxation is complete, the density matrix commutes with the total momentum. Averaging with any density matrix that has this symmetry would give zero for the correlation functions such as Eq. (7).

A very similar consideration can be applied to any quantum quench problem in which the symmetry of the Hamiltonian that governs the system's dynamics differs from that of the initial state. [For example, suddenly turned off interactions in the Luttinger model [10] correspond to the initial state with broken global U(1) symmetry]. The information about the symmetry of the initial state is encoded in the off-diagonal correlation functions [cf. Eq. (7)]; relaxation manifests itself in the decay of these off-diagonal correlations with time.

We now consider a specific but rather realistic situation when the trap potential Eq. (2) is harmonic,

$$V(x) = \frac{1}{2}\, m\omega^2 x^2 = \frac{x^2}{2ml^4}\,, \quad l = (m\omega)^{-1/2}. \tag{9}$$

At zero temperature the correlation function (7) can be written as

$$\langle \psi_k^\dagger \psi_{k'}\rangle_0 = \sum_{n=0}^{N-1} \phi_n^*(k)\phi_n(k'), \tag{10}$$

where $N \gg 1$ is the number of particles in the system and $\phi_n(k)$ is the stationary eigenfunction in the momentum representation that corresponds to n-th energy level of a harmonic oscillator. The expectation values of the fermionic occupation numbers (6) are obtained by setting $k = k'$ in Eq. (10). For $N \gg 1$, this yields a "semicircle" dependence

$$\langle n_k\rangle_0 = \frac{R}{\pi}\sqrt{1 - k^2/k_F^2}\,, \tag{11}$$

where k_F is the Fermi momentum and R is the classical radius of N-particle fermionic cloud confined in a harmonic trap,

$$k_F l = R/l = \sqrt{2N}.$$

In writing Eq. (11) we neglected the oscillating with k contribution that has a relative magnitude of the order of $l/R \ll 1$. This contribution is the momentum-space counterpart of the Friedel oscillations in $\langle \rho(x)\rangle_0$, as it is obvious from the operator identity [11]

$$n_k = \, e^{iH\tau}\left[l^2\rho(kl^2)\right] e^{-iH\tau}, \quad \tau = \frac{\pi}{2\omega}\,. \tag{12}$$

The period of the Friedel oscillations in x-space is the Fermi wavelength $2\pi/k_F$. Eq. (12) then implies that the corresponding oscillations in k-space have a period $2\pi/k_F l^2 = 2\pi/R$.

Although the Friedel oscillations contribute little to $\langle n_k\rangle_0$, they are responsible for the fluctuations of the fermionic occupation numbers. Indeed, using Eqs. (8) and (10), we find

$$\langle \delta n_k \delta n_{k'}\rangle_0 = -\,\frac{\sin^2\left[(k-k')R\right]}{\pi^2(k-k')^2}\,. \tag{13}$$

Eq. (13) is valid when both $|k|$ and $|k'|$ are small compared with the Fermi momentum k_F. In this limit the dependence of $\langle \delta n_k \delta n_{k'}\rangle_0$ on $k + k'$ [which we have neglected in writing Eq. (13)] is very weak.

The visibility of the Friedel oscillations Eq. (13) is not affected by temperature T as long as $T \ll \epsilon_F$, where $\epsilon_F = N\omega = k_F^2/2m$ is the Fermi energy (i.e., the chemical potential for hard-core bosons). Indeed, a finite temperature introduces an uncertainty $\delta R \sim RT/\epsilon_F$ in the size of the cloud which leads to the exponential decay of $\langle \delta n_k \delta n_{k'} \rangle_0$ at $|k - k'| \gtrsim 1/\delta R$. The oscillations (13) survive as long as $\delta R \ll R$, i.e., at $T \ll \epsilon_F$.

So far, we demonstrated that the information about the broken translational invariance in the initial state of the system is preserved in the statistics of the *fermionic* occupation numbers. In particular, it manifests itself in the characteristic oscillatory dependence of $\langle \delta n_k \delta n_{k'} \rangle_0$ on $k - k'$, see Eq. (13). However, the fermions emerged in our problem merely as a convenient way of dealing with the exact eigenstates of the system. Since the relation between the effective fermions and the original hard-core bosons is nonlocal, see Eq. (3), the behavior of the *bosonic* correlation functions is much more complex.

We discuss here the bosonic occupation numbers $f_k = \varphi_k^\dagger \varphi_k$. Unlike their fermionic counterparts Eqs. (6) and (8), the expectation value $\langle f_k \rangle_t$ (momentum distribution) and the fluctuations $\langle \delta f_k \delta f_{k'} \rangle_t$ (here $\delta f_k = f_k - \langle f_k \rangle_t$) are no longer constant. However, the time-dependent contributions are superpositions of an infinite number of oscillating terms and decay at $t \to \infty$ [12].

In order to find the bosonic occupations at $t \to \infty$, we again concentrate on the harmonic trap potential Eq. (9). Following the method of [13, 14, 15], we consider first the single-particle Schrödinger equation

$$i \frac{\partial}{\partial t} \psi_n(x,t) = -\frac{1}{2m} \frac{\partial^2}{\partial x^2} \psi_n(x,t)$$

with the initial condition $\psi_n(x, 0) = \phi_n(x)$, where $\phi_n(x)$ is the normalized stationary eigenfunction of a harmonic oscillator corresponding to the eigenenergy $\epsilon_n = (n + 1/2)\omega$. At $t \gg 1/\omega$, the wave function $\psi_n(x,t)$ assumes the form [16]

$$\psi_n(x,t) = \frac{1}{\sqrt{\omega t}} \exp\left(\frac{ix^2}{2l^2 \omega t} \right) e^{-i\epsilon_n \tau} \phi_n(x/\omega t) \,, \qquad (14)$$

where τ is given by Eq. (12). Upon introducing dimensionless variables

$$\eta = \omega t, \quad \xi = \frac{x}{\eta l} \,,$$

we rewrite Eq. (14) as

$$\psi_n(x,t) = \eta^{-1/2} e^{i\eta\xi^2/2} \, e^{-i\epsilon_n \tau} \phi_n(\xi l) \,. \qquad (15)$$

Using Eq. (15), the first-quantized many-particle wave function of hard-core bosons [17] Φ_t can be expressed via its initial value Φ_0,

$$\Phi_t\big(\{x_i\}\big) = \eta^{-N/2} e^{i\eta \sum \xi_i^2/2} e^{-iE_0\tau} \Phi_0\big(\{\xi_i l\}\big); \qquad (16)$$

here $\{x_i\} = x_1, \ldots, x_N$, $\xi_i = x_i/\eta l$, and $\Phi_0\big(\{x_i\}\big)$ is the many-body eigenstate of the initial Hamiltonian H with energy E_0.

In the second-quantized language, Eq. (16) implies the operator relation [11]

$$\varphi(x,t) = \eta^{-1/2} e^{i\eta\xi^2/2} \widetilde{\varphi}(\xi l, \tau), \qquad (17)$$

where $\varphi(x,t)$ and $\widetilde{\varphi}(x,t)$ are operators in the Heisenberg representation with the time dependence governed by the Hamiltonians H_0 and $H = H_0 + V$, respectively:

$$\varphi(x,t) = e^{iH_0 t} \varphi(x) \, e^{-iH_0 t}, \quad \widetilde{\varphi}(x,t) = e^{iHt} \varphi(x) \, e^{-iHt}.$$

Substitution of Eq. (17) into

$$f_k(t) = \frac{1}{2\pi} \int dx \, dx' \, e^{ik(x-x')} \varphi^\dagger(x,t) \varphi(x',t)$$

yields

$$\begin{aligned} f_k(t) \; &= \; \frac{l^2\eta}{2\pi} \int d\xi \, d\xi' e^{i\eta(\xi - \xi')\left[kl - (\xi+\xi')/2\right]} \\ &\qquad \times \widetilde{\varphi}^\dagger(\xi l, \tau) \widetilde{\varphi}(\xi' l, \tau). \end{aligned} \qquad (18)$$

At $\eta \to \infty$ the integral over ξ and ξ' here can be evaluated in the stationary phase approximation with the result [11]

$$f_k(t \to \infty) = e^{iH\tau} \big[l^2 \rho\big(kl^2\big) \big] e^{-iH\tau}. \qquad (19)$$

[Analogous calculation for fermions yields Eq. (12) which, unlike Eq. (19), is valid at all $t > 0$.] Finally, comparing Eq. (19) with Eq. (12), we find [11]

$$f_k(t \to \infty) = n_k. \qquad (20)$$

According to Eq. (20), at $t \to \infty$ the bosonic occupation numbers in k-space f_k coincide with the integrals of motion n_k. Since Eq. (20) holds for operators [11], it also implies that the statistical moments of the bosonic occupation numbers f_k at $t \to \infty$ coincide with those for fermions in the initial trapped state, e.g.,

$$\langle f_k \rangle_\infty = \langle n_k \rangle_0, \quad \langle \delta f_k \delta f_{k'} \rangle_\infty = \langle \delta n_k \delta n_{k'} \rangle_0. \qquad (21)$$

We pause now to discuss the conditions of applicability of Eqs. (20) and (21). Since we used the stationary phase method, the relevant characteristic time scales can be obtained by equating the scale of variation with ξ of the phase $\eta\xi^2$ in Eq. (17) with that of the field $\widetilde{\varphi}(\xi l, \tau)$.

The dependence of $\varphi(x)$ [and, therefore, of $\widetilde{\varphi}(x,\tau)$] on x is characterized by two length scales. The longer one is the size of the trapped system R. Neglecting all other scales, we find $t_1 \sim 1/\epsilon_F$; this corresponds to the time the particles in the trap move between the collisions (the mean free time). The shorter scale of variation of $\varphi(x)$ is the distance between particles R/N, which leads to the time scale $t_2 \sim 1/\omega \sim N t_1$. This is the time it takes for a particle moving with the Fermi velocity $v_F = k_F/m$

to cross the trap, $t_2 \sim R/v_F$, or, equivalently, for the expanding cloud to double its size.

Shortly after release of the trap, at $t_1 \ll t \ll t_2$, the oscillating transient contributions to the bosonic momentum distribution are still present, but $\langle f_k \rangle_t$ averaged over time is already given by the smooth "fermionic" semicircle Eq. (11), see [13]. In this regime the discreteness of the system is not important and the "shot noise" fluctuations, Eq. (13), are not yet resolved. Accordingly, the *hydrodynamic* description [13, 14] based on Eq. (11) provides a complete information about the system.

Much later, at $t \gg t_2$, the system enters the asymptotic regime where Eq. (20) is applicable. In this regime the transients have already decayed [12], the statistics of particles no longer matters, and statistical moments of the bosonic occupation numbers approach their steady-state values Eq. (21). In other words, this regime is essentially that of the *collisionless expansion* of the system.

It should be noted that the setup discussed here is essentially the same as that studied recently in [18]. Based on the behavior of $\langle f_k \rangle_t$, it was conjectured there that any isolated system with integrable dynamics *relaxes* to a state described by a certain generalized Gibbs distribution. According to the prescription of [18] adopted for continuously varying k [19], the density matrix at $t \to \infty$ has the form $\hat{\rho}_G \propto \exp\left(-\int dk\, \beta_k n_k\right)$. Although the coefficients β_k can always be chosen in such a way that $\mathrm{Tr}(\hat{\rho}_G n_k) = \langle n_k \rangle_0$ [18], finding β_k is obviously not sufficient to establish the validity of the conjecture. Indeed, no matter what β_k is, $\langle n_k n_{k'} \rangle = \mathrm{Tr}(\hat{\rho}_G n_k n_{k'}) \equiv \langle n_k \rangle \langle n_{k'} \rangle$, i.e., $\langle \delta n_k \delta n_{k'} \rangle = 0$. In view of Eqs. (13), (20), and (21), we conclude that $\hat{\rho}_G$ not only neglects the correlations between different integrals of motion n_k, but also fails to account for the correlations between the bosonic occupation numbers f_k. This raises serious doubts whether the generalized Gibbs distribution conjectured in [18] is actually useful for the description of the quantum quench problems.

Although our consideration relied rather heavily on the properties of the hard-core Bose gas with the harmonic initial confinement, we expect some of our conclusions to be generic. In particular, we expect that any finite 1D system with short-range interactions enters the collisionless expansion regime at $t \gg t_2 \sim R/v_s$, where v_s is the sound velocity in the initial trapped state. (In an infinite system, this time scale corresponds to the establishment of a local equilibrium in a subsystem of size R [6]).

In a nonintegrable system, a relaxation would occur at $t \lesssim t_2$. The relaxation would partially restore the translational invariance, leading to the suppression of the noise $\langle \delta f_k \delta f_{k'} \rangle_\infty$. Since the noise is not sensitive to temperature (see above), the accuracy of noise measurements in time-of-flight experiments is not limited by one's ability to control the temperature. This suggests that deviations from the integrability are easier to detect in noise measurements than, for example, by observing the saturation of the height of the peak in the dynamic structure factor with lowering the temperature [20] in Bragg spectroscopy experiments [4].

Finally, we emphasize that real-life 1D bosons are neither hard core nor their dynamics is integrable. It is conceivable that deviations from integrability will have a dramatic effect on the behavior of some observable quantities. Detailed understanding of the relaxation mechanisms and other consequences of nonintegrability in 1D Bose systems remains an important open problem.

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[1] I. Bloch, J. Dalibard, and W. Zwerger, arXiv:0704.3011.
[2] B. Paredes *et al.*, Nature **429**, 277 (2004); T. Kinoshita, T. Wenger, and D.S. Weiss, Science **305**, 1125 (2004).
[3] T. Kinoshita, T. Wenger, and D.S. Weiss, Nature **440**, 900 (2006).
[4] T. Stöferle *et al.*, Phys. Rev. Lett. **92**, 130403 (2004).
[5] B. Sutherland, *Beautiful Models* (World Scientific, Singapore, 2004).
[6] P. Calabrese and J. Cardy, J. Stat. Mech. P06008 (2007).
[7] E. Altman, E. Demler, and M.D. Lukin, Phys. Rev. A **70**, 013603 (2004).
[8] M. Greiner *et al.*, Phys. Rev. Lett. **94**, 110401 (2005); S. Fölling *et al.*, Nature **434**, 481 (2005).
[9] M.D. Girardeau, J. Math. Phys. **1**, 516 (1960).
[10] M.A. Cazalilla, Phys. Rev. Lett. **97**, 156403 (2006).
[11] The operator relations Eqs. (12), (17), (19), and (20) hold in the sense that the left- and the right-hand sides have the same matrix elements between any two many-particle eigenstates of the Hamiltonian (1) with harmonic confinement (2), (9).
[12] The decay of transients is analogous to the spreading of a wave packet in the single-particle quantum mechanics and should not be confused with relaxation.
[13] A. Minguzzi and D.M. Gangardt, Phys. Rev. Lett. **94**, 240404 (2005).
[14] B. Sutherland, Phys. Rev. Lett. **80**, 3678 (1998).
[15] Yu. Kagan, E.L. Surkov, and G.V. Shlyapnikov, Phys. Rev. A **54**, R1753 (1996).
[16] A.M. Perelomov and Ya.B. Zel'dovich, *Quantum Mechanics: Selected Topics* (World Scientific, Singapore, 1999).
[17] The many-particle wave function of hard-core bosons is given by $\Phi_t(\{x_i\}) = \Psi_t(\{x_i\}) \prod_{i<j} \mathrm{sign}(x_i - x_j)$. Here Ψ_t is the Slater determinant built of the single-particle wave functions $\psi_n(x,t)$ [9].
[18] M. Rigol *et al.*, Phys. Rev. Lett. **98**, 050405 (2007); M. Rigol, A. Muramatsu, and M. Olshanii, Phys. Rev. A **74**, 053616 (2006).
[19] The results (12), (13), (21) do not rely on the continuity of k and remain intact when the system expands in a finite box of the size $L \gg R$ rather than in a vacuum.
[20] M. Khodas *et al.*, Phys. Rev. Lett. **99**, 110405 (2007).